MINISTRY OF EDUCATION, SINGAPORE
in collaboration with
CAMBRIDGE ASSESSMENT INTERNATIONAL EDUCATION
General Certificate of Education Advanced Level
Higher 2

FURTHER MATHEMATICS 9649/01

Paper 1

For examination from 2025

SPECIMEN PAPER

3 hours

Additional Materials: Printed Answer Booklet
List of Formulae and Results (MF27)

READ THESE INSTRUCTIONS FIRST

Answer **all** questions.
Write your answers on the Printed Answer Booklet. Follow the instructions on the front cover of the answer booklet.

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are **not** allowed in a question, you must present the mathematical steps using mathematical notations and not calculator commands.
You must show all necessary working clearly.

The number of marks is given in brackets [ ] at the end of each question or part question.

Singapore Examinations and Assessment Board

Cambridge Assessment International Education

1 The curve with equation $y = e^{\frac{1}{2}x} + e^{-\frac{1}{2}x}$, for $0 \leqslant x \leqslant \ln 2$, is rotated through 2π radians about the x-axis.

Find the **exact** value of the surface area generated. [5]

2 The sequence $\{u_n\}$ is given by $u_1 = 2$ and $u_{n+1} = \dfrac{12}{k - u_n}$ $(n \geqslant 1)$, where k is a given real constant.

Use your graphing calculator to describe the behaviour of $\{u_n\}$ in each of the cases:

- $k = 8$,
- $k = 7$,
- $k = 6$,
- $k = 5$. [6]

3 The equation $f(x) = 0$ has a root α. Various numerical methods which are used to determine the value of α (to a suitable degree of accuracy) generate a sequence of approximations, $\{x_n\}$, starting with an initial approximation, x_0. One such method is *Halley's method*, with iteration formula

$$x_{n+1} = x_n - \frac{2f(x_n)f'(x_n)}{2\left[f'(x_n)\right]^2 - f(x_n)f''(x_n)}.$$

(a) The *Newton-Raphson method* and *Halley's method* are both used to find approximations to the cube root of 2, using $f(x) = x^3 - 2$ and $x_0 = 1$. For each method, determine the values of x_1 and x_2, giving each answer correct to 4 decimal places. [5]

(b) By comparing the iteration formulae for these two methods, state the conditions under which the two methods would give approximately equal values of x_{n+1} for a given x_n. [2]

4 The area bounded by the curve with equation $y = x + 2\sin x$, for $0 \leqslant x \leqslant 4\pi$, and the x-axis is rotated through one full turn about the y-axis. The volume generated is denoted by V.

(a) With the aid of a sketch graph, explain why it is more appropriate to determine the exact value of V using the shell method (rather than the disc method). [2]

(b) Use the shell method to determine the exact value of V. [5]

5 The surface S has equation $z = \mathrm{f}(x, y)$, where $\mathrm{f}(x, y) = \dfrac{4x}{y} + \dfrac{y^2}{2x} + 3xy$, for $x, y > 0$. The point A on S has coordinates $(1, 2, 10)$.

(a) Determine the values of f_x, f_y, f_{xx}, f_{yy}, f_{xy} and f_{yx} at A. [7]

The quadratic approximation for S in the region of A is given by $z = Q(x, y)$.

(b) Show that $Q(x, y)$ can be written in the form $2(x + \frac{1}{2})^2 + y^2 + k$, where k is a rational number to be determined. [3]

6 **(a)** **(i)** Solve the equation $z^7 = \mathrm{i}$, giving your answers in the form $r\,\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$. [3]

(ii) Hence, or otherwise, solve the equation $\left(\dfrac{z}{\sqrt{3} + \mathrm{i}}\right)^7 = \mathrm{i}$, giving your answers in a similar form. [3]

(b) **(i)** On the same Argand diagram, sketch the loci of points given by each of the following equations:

$$L_1\colon \left|z - \left(\sqrt{3} + 2\mathrm{i}\right)\right| = 2,$$

$$L_2\colon \arg\left(z - \sqrt{3}\right) = \frac{\pi}{3}.$$

[3]

(ii) Find, in the form $x + \mathrm{i}y$, the complex number which represents the point in the Argand diagram which is on both L_1 and L_2. [2]

7 The curve C has polar equation $r = 2(1 - \cos\theta)$, $0 \leqslant \theta \leqslant 2\pi$.

(a) Sketch C. [1]

(b) Find the total length of C. [5]

(c) By considering the curve D with polar equation $r = 2(1 - \sin\theta)$, $0 \leqslant \theta \leqslant 2\pi$, determine the exact value of $\displaystyle\int_0^{2\pi} \sqrt{1 - \sin x}\,\mathrm{d}x$. [5]

8 The matrices **A** and **B** are such that $\mathbf{A} = \begin{pmatrix} k & 4 & 3 \\ 1 & 0 & -2 \\ -2 & -1 & k \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -2 & -11 & -8 \\ k & 10 & 7 \\ -1 & -6 & -4 \end{pmatrix}$.

(a) **(i)** Determine **AB**. [3]

(ii) Show that, when $k = 2$, $\mathbf{B} = \mathbf{A}^{-1}$. [1]

The matrix **M** has eigenvalues $\lambda_1 = 1$, $\lambda_2 = -1$ and $\lambda_3 = 2$, with corresponding eigenvectors $\mathbf{u}_1 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$, $\mathbf{u}_2 = \begin{pmatrix} 4 \\ 0 \\ -1 \end{pmatrix}$ and $\mathbf{u}_3 = \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix}$ respectively.

(b) **(i)** State, with justification, whether $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$ form a basis for the space of column vectors of the form $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$, where a, b and c are real. [1]

(ii) Given that the vector $\mathbf{v} = \begin{pmatrix} 1 \\ 14 \\ -20 \end{pmatrix}$ can be expressed in the form $\mathbf{v} = 4\mathbf{u}_1 + 2\mathbf{u}_2 - 5\mathbf{u}_3$, evaluate $\mathbf{M}^7\mathbf{v}$ without calculating any power of **M**. [3]

(c) Showing all necessary working, find **M** as a single 3×3 matrix. [4]

9 The sequence $\{X_n\}$ is defined by

$$X_0 = \frac{1}{16},\ X_1 = \frac{17}{16} \text{ and } X_{n+1} = 34X_n - X_{n-1} \text{ for } n \geqslant 1.$$

(a) Determine the solution of this second-order recurrence system. [5]

The sequence $\{Y_n\}$ is defined by $Y_n = \sqrt{X_n - \frac{1}{16}}$ for all $n \geqslant 0$,

(b) Calculate the values of Y_n for $n = 0$ to 4. [2]

(c) **(i)** It is given that the sequence $\{Y_n\}$ satisfies a second-order recurrence relation. Use your answers to part **(b)** to write this recurrence relation down. [1]

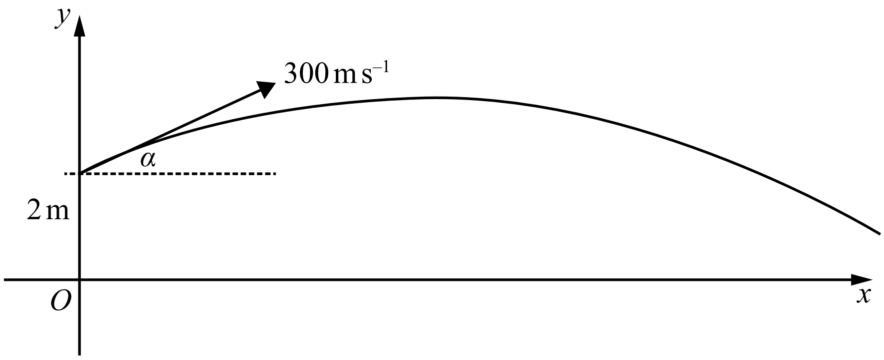
(ii) Write down, in the form $a + b\sqrt{2}$, where a and b are integers, the positive square-root of $17 + 12\sqrt{2}$. [1]

(iii) Hence, without solving the recurrence system for $\{Y_n\}$, and in either order:

- find the solution for Y_n as a function of n;
- show that $\left(X_n - \frac{1}{16}\right)$ is the square of an integer for all integers $n \geqslant 0$. [5]

10 [In this question, all variables are in standard S.I. units.]

An object P of constant mass m moves in a vertical plane. At time t, its displacement from the origin O is given by the vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$, where x and y are P's linear displacements in the directions of the horizontal and vertical axes through O, as shown on the diagram below.
Also, at time t, the velocity of P is $\mathbf{v} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j}$ and its acceleration is $\mathbf{a} = \ddot{x}\mathbf{i} + \ddot{y}\mathbf{j}$.



At time $t = 0$, P is projected from the point 2 m directly above O with speed $300\,\mathrm{m\,s^{-1}}$ at an angle $\alpha = \tan^{-1}\frac{7}{24}$ above the horizontal.

(a) State the initial position vector, $\mathbf{r}_0$, of P, and explain why $\mathbf{v}_0$, the initial velocity vector of P, is given by $\mathbf{v}_0 = 288\mathbf{i} + 84\mathbf{j}$. [1]

Newton's Second Law states that the vector sum of all of the forces that act on P is equal to the product $m\mathbf{a}$. [The sign of each force represents its direction of application.]

The subsequent motion of P is modelled in the following way. There are only three forces acting on P:

- its weight, of magnitude mg, acting vertically downwards (where $g = 9.8\,\mathrm{m\,s^{-2}}$, is the acceleration due to gravity)
- a resistive force, $\mathbf{R}$, for positive constant k, which directly opposes the motion of P and is proportional to the velocity $\mathbf{v}$
- a second resistive force, $\mathbf{S}$, due to the wind which is blowing horizontally in the negative x direction. $\mathbf{S}$ is taken to be proportional to the horizontal displacement of P from O.

(b) Use Newton's Second Law to justify the statement that P's motion is described by the vector equation $\mathbf{a} + k\mathbf{v} = -lx\mathbf{i} - g\mathbf{j}$, for positive constants k and l. [2]

(c) Given that $k = 1.2$ and $l = 0.2$,

(i) write down the differential equation that governs the motion of P in the positive x direction, and hence show that $x = 360(\mathrm{e}^{-0.2t} - \mathrm{e}^{-t})$; [4]

(ii) write down the differential equation that governs the motion of P in the positive y direction, and hence determine the horizontal displacement of P at the instant when it lands on the x-axis. [10]

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